EECE 340 SUMMER 2011
QUIZ I

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PROBLEM 1

a) $y(n) = x(n-1) + |n\,x(n)|$

This system is not time invariant

Let $x_1(n)$ be an input with $y_1(n)$ corresponding output

$$y_1(n) = x_1(n-1) + |n\,x_1(n)|$$

Let $x_2(n) = x_1[n-n_0]$

Then $y_2(n) = x_2(n-1) + |n\,x_2(n)|$

$$= x_1(n-n_0-1) + |n\,x_1(n-n_0)|$$

but $y_1(n-n_0) = x_1(n-n_0-1) + |(n-n_0)\,x_1(n-n_0)|$

Thus $y_2(n) \neq y_1(n-n_0) \Rightarrow$ NOT T.I.

It is also not linear

let $x_1(n) = -1 \;\forall n \qquad x_2(n) = 2 \;\forall n$

$y_1(n) = -1 + |n(-1)| = n-1$

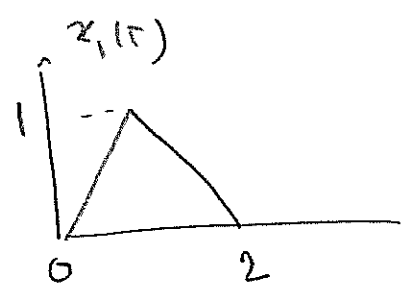
$y_2(n) = 2 + |n(2)| = 2+2n$

if $x_3(n) = x_1(n) + x_2(n) = 1$, Then $y_3(n) = 1 + |n(1)| = n+1$

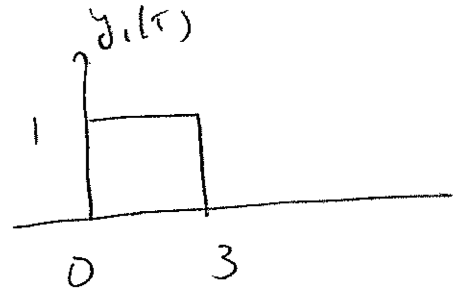
but $y_1(n) + y_2(n) = 3n+1 \neq y_3(n)$

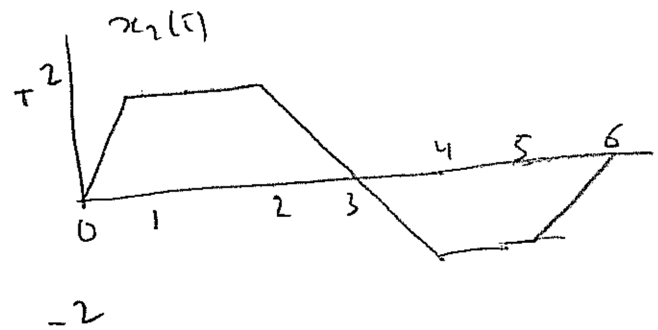
This is also obvious if you scale the input by a negative constant (absolute value is not linear)

5)



$\Rightarrow$





$\Rightarrow$?

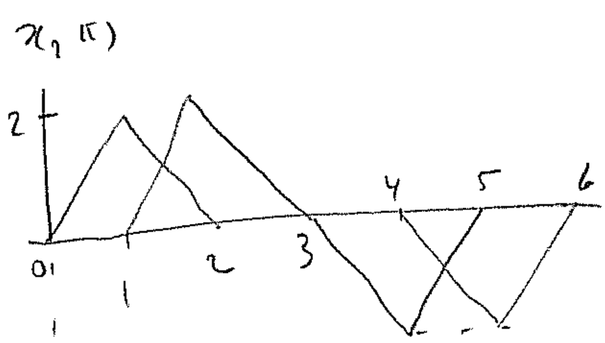
Note that $x_2(t)$ can be written in terms of $x_1(t)$

$$x_2(t) = 2\left[x_1(t) + x_1(t-1) - x_1(t-3) - x_1(t-4)\right]$$

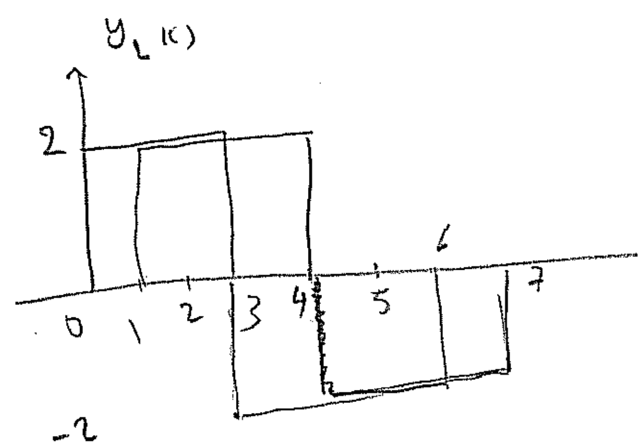
Thus since system is LTI

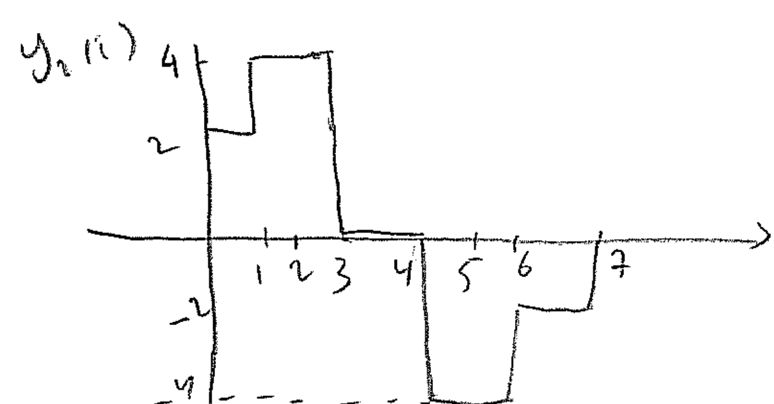
$$y_2(t) = 2\left[y_1(t) + y_1(t-1) - y_1(t-3) - y_1(t-4)\right]$$

Sketching:



$\Rightarrow$





c) $y(n) - \frac{1}{4} y(n-1) - \frac{1}{8} y[n-2] = 4x(n) + \frac{1}{2} x(n-2)$

i) ZSR : find $H(z)$

$$\left(1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}\right) Y(z) = \left(4 + \frac{1}{2}z^{-2}\right) X(z)$$

$$\Rightarrow H(z) = \frac{4 + \frac{1}{2}z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$$

for $x(n) = \left(\frac{1}{2}\right)^n u(n) \Rightarrow X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$

$$Y(z) = \frac{\left(4 + \frac{1}{2}z^{-2}\right)}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{4}z^{-1}\right)} \cdot \frac{1}{1 - \frac{1}{2}z^{-1}}$$

Note $H(z)$ is not proper, so it can be simplified

$$\begin{array}{r|l} & -4 \\ \hline -\frac{1}{8}\lambda^2 - \frac{1}{4}\lambda + 1 & \frac{1}{2}\lambda^2 + 4 \\ & \frac{1}{2}\lambda^2 + \lambda - 4 \\ \hline & 8 - \lambda \end{array}$$

$$\Rightarrow H(z) = -4 + \frac{8 - z^{-1}}{\left(1 + \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}$$

$$\therefore Y(z) = \frac{-4}{1 - \frac{1}{2}z^{-1}} + \frac{8 - z^{-1}}{\left(1 + \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)^2}$$

$$Y(z) = \frac{4 + \frac{1}{2}\lambda^2}{\left(1 - \frac{1}{2}\lambda\right)^2\left(1 + \frac{1}{4}\lambda\right)} = \frac{8\lambda^2 + 64}{(\lambda - 2)^2(\lambda + 4)}$$

performing P.F.E (have to rewrite with λ having a multiplier of one)

$$Y(\lambda) = \frac{A}{(\lambda-2)^2} + \frac{B}{\lambda-2} + \frac{C}{\lambda+4}$$

$$A = \left.\frac{8\lambda^2+64}{\lambda+4}\right|_{\lambda=2} = \frac{96}{6} = 16$$

$$B = \left.\frac{d}{d\lambda}\left(\frac{8\lambda^2+64}{\lambda+4}\right)\right|_{\lambda=2} = \left.16\lambda\frac{(\lambda+4)-8\lambda^2-64}{(\lambda+4)^2}\right|_{\lambda=2} = \frac{96}{36} = \frac{16}{6} = \frac{8}{3}$$

$$C = \left.\frac{8\lambda^2+64}{(\lambda-2)^2}\right|_{\lambda=-4} = \frac{192}{36} = \frac{16}{3}$$

$$\therefore Y(\lambda) = \frac{16}{(\lambda-2)^2} + \frac{\frac{8}{3}}{\lambda-2} + \frac{\frac{16}{3}}{\lambda+4}$$ where $\lambda = z^{-1}$

$$= \frac{16}{4(1-\frac{1}{2}z^{-1})^2} - \frac{4}{3(1-\frac{1}{2}z^{-1})} + \frac{\frac{4}{3}}{1+\frac{1}{4}z^{-1}}$$

$$= 4(n+1)\left(\tfrac{1}{2}\right)^{n+1}u(n+1) - \frac{4}{3}\left(\tfrac{1}{2}\right)^n u(n) + \frac{4}{3}\left(-\tfrac{1}{4}\right)^n u(n)$$

$$= 2(n+1)\left(\tfrac{1}{2}\right)^n - \frac{4}{3}\left(\tfrac{1}{2}\right)^n + \frac{4}{3}\left(-\tfrac{1}{4}\right)^n \qquad n \geq 0$$

$$= \frac{2}{3}\left(\tfrac{1}{2}\right)^n + 2n\left(\tfrac{1}{2}\right)^n + \frac{4}{3}\left(-\tfrac{1}{4}\right)^n \qquad n \geq 0$$

if input is $x(n) = (-1)^n u(n)$

$\Rightarrow X(z) = \dfrac{1}{1+z^{-1}}$

$$Y(z) = H(z) \cdot X(z) = \frac{4+\frac{1}{2}z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)\left(1+z^{-1}\right)}$$

$$= \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+\frac{1}{4}z^{-1}} + \frac{C}{1+z^{-1}}$$

$$A = \left.\frac{4+\frac{1}{2}z^{-2}}{\left(1+\frac{1}{4}z^{-1}\right)\left(1+z^{-1}\right)}\right|_{z^{-1}=2} = \frac{6}{\left(\frac{3}{2}\right)(3)} = \frac{4}{3}$$

$$B = \left.\frac{4+\frac{1}{2}z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+z^{-1}\right)}\right|_{z^{-1}=-4} = \frac{12}{(3)(-3)} = -\frac{4}{3}$$

$$C = \left.\frac{4+\frac{1}{2}z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}\right|_{z^{-1}=-1} = \frac{\frac{9}{2}}{\left(\frac{3}{2}\right)\left(\frac{3}{4}\right)} = 4$$

$$Y(z) = \frac{\frac{4}{3}}{1-\frac{1}{2}z^{-1}} - \frac{\frac{4}{3}}{1+\frac{1}{4}z^{-1}} + \frac{4}{1+z^{-1}}$$

$$y(n) = \frac{4}{3}\left(\frac{1}{2}\right)^n - \frac{4}{3}\left(-\frac{1}{4}\right)^n + 4(-1)^n \qquad n \geq 0$$

$$H(z) = -4 + \frac{3 - z^{-1}}{\left(1+\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$$

$$= -4 + \frac{A}{1+\frac{1}{4}z^{-1}} + \frac{B}{1-\frac{1}{2}z^{-1}}$$

$$\therefore A = \left.\frac{3-z^{-1}}{1-\frac{1}{2}z^{-1}}\right|_{z^{-1}=-4} = \frac{12}{3} = 4$$

$$B = \left.\frac{3-z^{-1}}{1+\frac{1}{4}z^{-1}}\right|_{z^{-1}=2} = \frac{6}{\frac{3}{2}} = 4$$

$$= -4 + \frac{4}{1+\frac{1}{4}z^{-1}} + \frac{4}{1-\frac{1}{2}z^{-1}}$$

$$h(n) = -4\delta(n) + 4\left(-\frac{1}{4}\right)^n u(n) + 4\left(\frac{1}{2}\right)^n u(n)$$

since system is causal & poles inside unit circle $\Rightarrow$ stable.

d) input $u(t)$ $\longrightarrow$ $y(t) = \alpha e^{-4t} + \beta e^{-5t}$ $t \geq 0$

input $e^{t}u(t)$ $\longrightarrow$ $y(t) = 2e^{-t} + \gamma e^{-4t} + \theta e^{-5t}$ $t \geq 0$

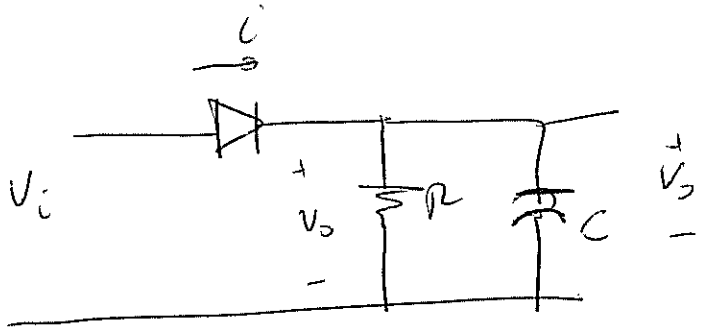
- The terms e^{-4t}, e^{-5t} are transients due to characteristic poly'l (natural frequencies).
- The input $u(t)$ does not show in output $\Rightarrow$ Transfer function has a zero at $s=0$ $\left(H(s)\big|_{s=0} = 0\right)$
- $H(s)\big|_{s=-1} = 2$

$$\therefore H(s) = \frac{Ks}{(s+4)(s+5)} \quad \& \quad \frac{K(-1)}{(-1+4)(-1+5)} = 2$$

$$\Rightarrow K = -24$$

$$H(s) = \frac{-24s}{(s+4)(s+5)}$$

c)



Circuit is not linear since diode has the characteristic $i = 0$ if $V_o \geq V_i$

$i \neq 0$ if $V_o < V_i$

To see this, apply DC source $V_i = +1\ V\,dc$

Then $V_o = 1\ V\,dc$

apply DC source $V_i' = -1\ V\,dc$

$V_o' = 0\ V\,dc \neq -1 \times V_o$

b) $H(s) = \dfrac{s+2}{s^2+s} = \dfrac{s+2}{s(s+1)}$

1- poles are at $s=0$, $s=-1$, system is not stable

2- $x(t) = 4e^{-t}u(t)$

$Y_{ZSR}(s) = H(s) \cdot X(s) = \dfrac{4}{s+1} \cdot \dfrac{s+2}{s(s+1)}$

$Y_{ZSR}(s) = \dfrac{4(s+2)}{s(s+1)^2} = \dfrac{A}{s} + \dfrac{B}{s+1} + \dfrac{C}{(s+1)^2}$

$A = \left.\dfrac{4(s+2)}{(s+1)^2}\right|_{s=0} = 8$

$B = \left.\dfrac{d}{ds}\dfrac{4(s+2)}{s}\right|_{s=-1} = \left.\dfrac{4s-4s-8}{s^2}\right|_{s=-1} = -8$

$C = \left.\dfrac{4(s+2)}{s}\right|_{s=-1} = -4$

$$\therefore Y_{ZSR}(s) = \frac{8}{s} \quad \frac{-8}{s+1} \quad \frac{-4}{(s+1)^2}$$

$$\text{or } y_{ZSR}(t) = \left(8 - 8e^{-t} - 4te^{-t}\right) u(t)$$

B-3) $y(0) = y'(0) = 0$

$$y_{TOT}(t) = y_{ZSR}(t) + y_{ZIR}(t)$$

$$y_{ZIR}(t) = \left(\alpha e^{0t} + \beta e^{-t}\right) u(t)$$

$$\therefore y_{TOT}(t) = \left(8 - 8e^{-t} - 4te^{-t} + \alpha + \beta e^{-t}\right) u(t)$$

$$y_{TOT}(0) = 0 = 8 - 8 + \alpha + \beta \Rightarrow \alpha + \beta = 0$$

$$y'_{TOT}(0) = 0 = 8 + 8e^{-t} - 4e^{-t} + 4te^{-t} - \beta e^{-t}\Big|_{t=0} = 8 + 8 - 4 - \beta = 0$$

$$\Rightarrow \beta = 12$$

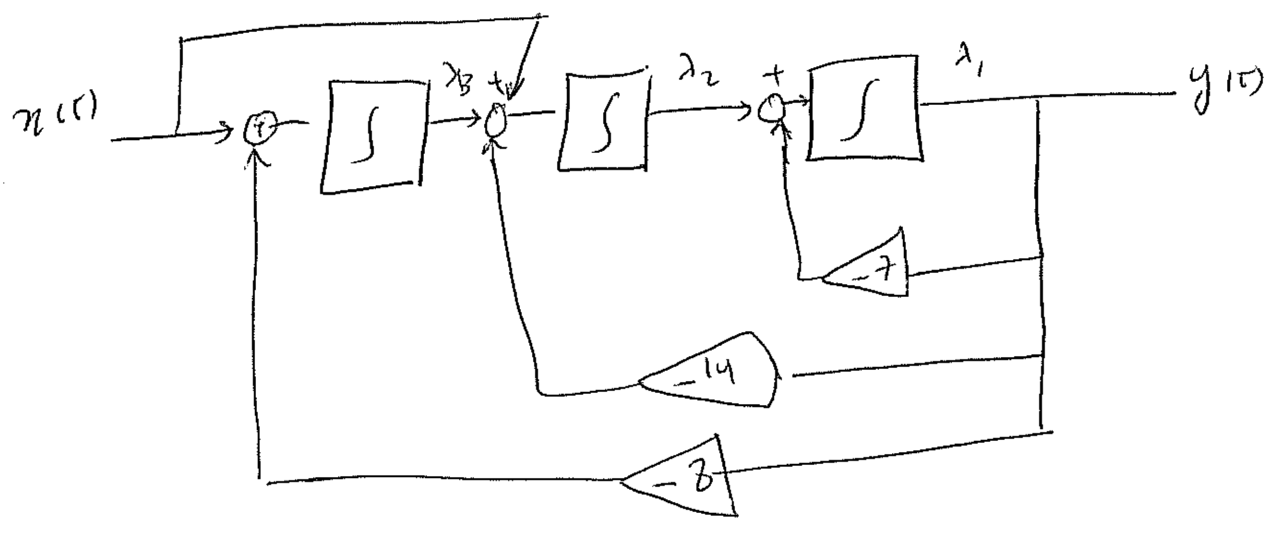
$$\alpha = -12$$

$$\therefore y_{TOT}(t) = -4 + 4e^{-t} - 4te^{-t} \quad t \geq 0$$

PROBLEM 2

$$\frac{d^3y}{dt^3} + 7\frac{d^2y}{dt^2} + 14\frac{dy}{dt} + 8y = x + \frac{dx}{dt}$$

a) rewrite as $y(t) = -7\int y(t)dt - 14\iint y(t)dt$
$- 8\iiint y(t)dt + \iiint x(t)dt + \iint x(t)dt$



$\dot{\lambda}_1 = -7\lambda_1 + \lambda_2$
$\dot{\lambda}_2 = -14\lambda_1 + \lambda_3 + x$
$\dot{\lambda}_3 = -8\lambda_1 + x$
$y = \lambda_1$

OR

$$\begin{bmatrix}\dot{\lambda}_1\\ \dot{\lambda}_2\\ \dot{\lambda}_3\end{bmatrix} = \begin{bmatrix}-7 & 1 & 0\\ -14 & 0 & 1\\ -8 & 0 & 0\end{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix} + \begin{bmatrix}0\\ 1\\ 1\end{bmatrix}x(t)$$

$$y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix} + \begin{bmatrix}0\end{bmatrix}.x(t)$$

b) $x(t) = \cos t\, u(t)$, at steady state
$y(t) = |H(j1)| \cos(t + \angle H(j1))$
as long as system is stable.

$$H(s) = \frac{s+1}{s^3 + 7s^2 + 14s + 8}$$

check for stability using RH criterion:

s^3	1	14
s^2	7	8
s	$\frac{98-8}{7}$	
1	1	

$\Rightarrow$ No sign changes $\Rightarrow$ stable.

$$H(j1) = \left.\frac{s+1}{s^3+7s^2+14s+8}\right|_{s=j1}$$

$$= \frac{j1+1}{-j-7+14j+8} = \frac{1+j1}{1+j13}$$

$$|H(j1)| = \frac{\sqrt{2}}{\sqrt{170}} = \frac{1}{\sqrt{85}} \approx$$

$$\angle H(j1) = 45^\circ - \tan^{-1}(13)$$

$\therefore$ for a cosine signal of amplitude 1 and freq 1 rad/sec

we expect the output to be of amplitude $\frac{1}{\sqrt{85}}$

and shifted by $\approx$ $^\circ$

PROBLEM 3

$$w[n+2] = w[n+1] - \frac{N}{50} w(n) + \frac{C}{250}$$

a) if $C = 1500$ $\quad n \geq 0$

$$\Rightarrow w[n+2] = w(n+1) - \frac{N}{50} w(n) + 6$$

need $\displaystyle w(n)_{n\to\infty} = 75$ for an input of 6 $n \geq 0$ or $6u(n)$

Apply Z-Transform to determine the system function

$$\left(z^2 - z + \frac{N}{50}\right) W(z) = \frac{6}{1-z^{-1}} = X(z)$$

where $\alpha = \frac{N}{50}$

$$\Rightarrow H(z) = \frac{1}{z^2 - z + \alpha}$$

for $X(z) = \frac{6}{1-z^{-1}}$, Then $W(z) = H(z) \cdot X(z)$

$$= \frac{6}{z^2 - z + \alpha} \cdot \frac{1}{1-z^{-1}}$$

which can be solved to find $w(n)$ and evaluated at $n \to \infty$

alternatively, since input is of the form $a^n u(n)$ where $a = 1$,

Then output at steady state will be $H(z)\Big|_{z=1} \cdot 6 \cdot 1^n u(n)$

$$\therefore y(n) = 6 \cdot H(z=1) = 75$$

$$\Rightarrow \frac{6}{1-1+\alpha} = 75 \Rightarrow \alpha = \frac{6}{75} \Rightarrow \frac{N}{50} = \frac{6}{75}$$

$$\Rightarrow N = \frac{300}{75} = 4 \text{ laps}$$

Note that $H(z)$ becomes $H(z) = \frac{1}{z^2 - z + \frac{6}{75}}$ with poles inside unit circle $\Rightarrow$ stable.

b) if $C = 3500$, Then

$$w(n+2) = w(n+1) - \frac{N}{50} w(n) + \frac{3500}{250}$$

again $\displaystyle \lim_{n\to\infty} y(n) = \frac{3500}{250} \cdot H(z)\Big|_{z=1}$

$$\Rightarrow \frac{3500}{250} \cdot \frac{1}{1-1+\alpha} = 75$$

$$\Rightarrow \alpha = \frac{3500}{(250)(75)} \Rightarrow \frac{N}{50} = \frac{3500}{(250)(75)}$$

$$\Rightarrow N = \frac{3500}{(5)(75)} = 9.3 \text{ laps}$$

In This Case $\alpha = 0.187$ which has

$$\Rightarrow H(z) = \frac{1}{z^2 - z + 0.187}$$

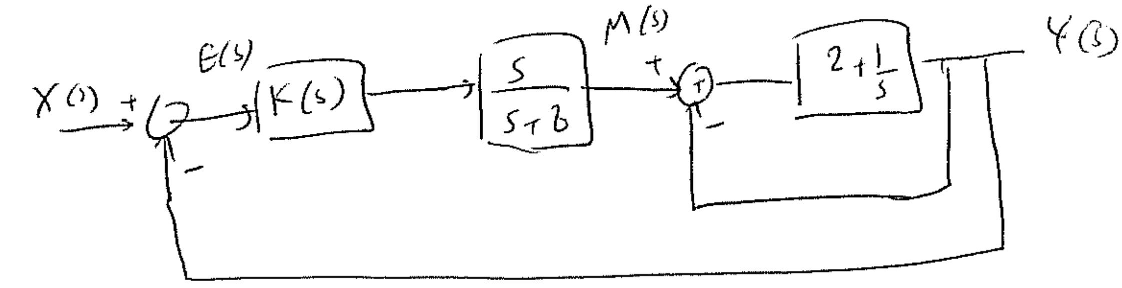
poles at $z_{1,2} = \dfrac{1 \pm \sqrt{1 - 0.747}}{2}$

$= 0.752, -0.248$

poles inside unit disc, so system is stable.

$\Rightarrow$ still achievable

PROBLEM 4



a)

$$\frac{Y(s)}{M(s)} = \frac{2+\frac{1}{s}}{1+2+\frac{1}{s}} = \frac{2s+1}{3s+1}$$

$$\frac{Y(s)}{E(s)} = \left(\frac{2s+1}{3s+1}\right)\left(\frac{s}{s+8}\right)\left(K(s)\right) = \frac{K_0(2s+1)s}{(s+8)(3s+1)}$$

$$\therefore \frac{Y(s)}{X(s)} = \frac{\frac{K_0 s(2s+1)}{(s+8)(3s+1)}}{1+\frac{K_0 s(2s+1)}{(s+8)(3s+1)}} = \frac{K_0 s(2s+1)}{3s^2+25s+8+2K_0 s^2+K_0 s}$$

$$= \frac{K_0 s(2s+1)}{(3+2K_0)s^2+(25+K_0)s+8}$$

for stability $3+2K_0 > 0$ & $25+K_0 > 0$

$\Rightarrow K_0 > -\frac{3}{2}$ & $K_0 > -25$

$\Rightarrow \boxed{K_0 > -\frac{3}{2}}$

b) for unit step input, we need Tracking.

for a general controller $K(s)$

$$\frac{Y(s)}{X(s)} = \frac{K(s)\cdot s(2s+1)/(s+8)(3s+1)}{1+\frac{K_0(s)\cdot s(2s+1)}{(s+8)(3s+1)}} = H(s)$$

if $X(s) = \frac{1}{s}$ (unit step)

$$Y(s) = H(s)\cdot X(s)$$

& $y(t)\Big|_{t\to\infty} = \lim_{s\to 0} s\cdot Y(s) = \lim_{s\to 0} sH(s)\cdot\frac{1}{s} = \lim_{s\to 0} H(s)$

as long $H(s)$ has no poles outside LHP

Alternatively look at the error $E(s)$

$$\frac{E(s)}{X(s)} = \frac{1}{1+\frac{K_0(s)\cdot s(2s+1)}{(s+8)(3s+1)}} = H_1(s)$$

$\Rightarrow E(s) = H_1(s)\cdot X(s)$

and $\displaystyle e(t)_{t\to\infty} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} H_1(s)$

I need $\displaystyle e(t)_{t\to\infty} = e_{ss} = 0 \Rightarrow \lim_{s\to 0} H_1(s) = 0$

if $K(s) = K_0$, then $H_1(s) = \dfrac{(s+8)(3s+1)}{(3+2K_0)s^2+(25+K_0)s+8}$

& $e_{ss} = 1$. $\Rightarrow$ no Tracking.

ii) if $K(s) = K_0 + K_1 s$

Then $H_1(s) = \dfrac{(s+8)(3s+1)}{3s^2 + 25s + 8 + (K_0 + K_1 s)(s)(2s+1)}$

$= \dfrac{(s+8)(3s+1)}{3s^2 + 25s + 8 + 2K_0 s^2 + K_0 s + 2K_1 s^3 + K_1 s^2}$

which again can only go to 1 as $s \to 0$
(assuming stable).

iii) $K(s) = K_0 + \dfrac{K_1}{s^2}$

Then $H_1(s) = \dfrac{(s+8)(3s+1)}{3s^2 + 25s + 8 + \left(K_0 + \frac{K_1}{s^2}\right)(s)(2s+1)}$

$= \dfrac{s(s+8)(3s+1)}{3s^3 + 25s^2 + 8s + (K_0 s^2 + K_1)(2s+1)}$

$= \dfrac{s(s+8)(3s+1)}{(3+2K_0)s^3 + (25+K_0)s^2 + (8+2K_1)s + K_1}$

$e_{ss} = \lim_{s \to 0} H_1(s) \cdot s \cdot \frac{1}{s} = H_1(s)\big|_{s=0} = 0$

as long as stability is preserved (not required to find ranges...)